

Chapter 4 Integration

§ One variable integral

$$\int_a^b f(x)dx$$

Approximate $f(x)$ by Lagrange's polynomial $P_n(x)$:

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} (x-x_0)\dots(x-x_n)$$

$$\int_a^b f(x)dx = \int_a^b P_n(x)dx + \frac{1}{(n+1)!} \int_a^b f^{(n+1)}(\xi(x))(x-x_0)\dots(x-x_n)dx$$

$$\int_a^b f(x)dx \approx \int_a^b P_n(x)dx$$

1. Trapezoidal rule

$$P_1(x) = f(x_0) \frac{x-x_1}{x_0-x_1} + f(x_1) \frac{x-x_0}{x_1-x_0}$$

Let $h = x_1 - x_0$, $sh = x - x_0$

$$P_1(x_0 + sh) = f(x_0)(1-s) + f(x_1)s$$

$$\int_{x_0}^{x_1} f(x)dx = \frac{h}{2} [f(x_0) + f(x_1)] - \frac{h^3}{12} f''(\xi)$$

2. Simpson's rule

a. $n = 2$

$$P_2(x) = f(x_0) \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1) \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2) \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$P_2(x) = f(x_0) \frac{(s-1)(s-2)}{2} + f(x_1) \frac{s(s-2)}{-1} + f(x_2) \frac{s(s-1)}{2}$$

$$\int_{x_0}^{x_2} f(x)dx = \frac{1}{3} h [f(x_0) + 4f(x_1) + f(x_2)] - \frac{h^5}{90} f^{iv}(\xi)$$

b. $n = 3$

$$\int_{x_0}^{x_3} f(x)dx = \frac{3}{8} h [f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)] - \frac{3h^5}{90} f^{iv}(\xi)$$

c. $n = 4$

$$\int_{x_0}^{x_4} f(x)dx = \frac{2h}{45} [7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)] - \frac{8h^7}{945} f^{iv}(\xi)$$

3. Composite rule

$$x_0 = a, \quad x_{2n} = b, \quad h = (b-a)/2n, \quad x_i = x_0 + ih$$

a. Composite Trapezoidal rule

$$\int_a^b f(x)dx = \frac{h}{2}[f(a) + 2\sum_{i=1}^{2n-1} f(x_i) + f(b)] - \frac{b-a}{12}h^2 f''(\xi)$$

b. Composite Midpoint rule

$$\int_a^b f(x)dx = 2h[f(x_1) + f(x_3) + \dots + f(x_{2n-1})] + \frac{b-a}{6}h^2 f''(\xi)$$

c. Composite Simpson's rule

$$\begin{aligned} \int_a^b f(x)dx &= \int_{x_0}^{x_{2n}} f(x)dx = \sum_{i=1}^n \int_{x_{2(i-1)}}^{x_{2i}} f(x)dx = \sum_{i=1}^n \frac{h}{3}[f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})] \\ &= \frac{h}{3}[f(x_0) + 2\sum_{i=2}^n f(x_{2i-2}) + 4\sum_{i=1}^n f(x_{2i-1}) + f(x_{2n})] - \frac{b-a}{180}h^4 f^{iv}(\xi) \end{aligned}$$

4. Gaussian Quadrature

$$\int_{-1}^1 f(x)dx = \sum_{i=1}^n c_i f(x_i)$$

Question: How to choose c_i and x_i such that

$$\min[\int_{-1}^1 f(x)dx - \sum_{i=1}^n c_i f(x_i)]^2$$

Ex. Four parameters x_1, x_2, c_1 and c_2 generate a polynomial of order 3

with the basis $\{1, x, x^2, x^3\}$.

$$\int_{-1}^1 1dx = c_1 \cdot 1 + c_2 \cdot 1, \quad \int_{-1}^1 xdx = c_1 \cdot x_1 + c_2 \cdot x_2, \quad \int_{-1}^1 x^2dx = c_1 \cdot x_1^2 + c_2 \cdot x_2^2,$$

$$\int_{-1}^1 x^3dx = c_1 \cdot x_1^3 + c_2 \cdot x_2^3$$

$$\text{Yield } x_1 = -\frac{\sqrt{3}}{3}, \quad x_2 = \frac{\sqrt{3}}{3}, \quad c_1 = c_2 = 1.$$

Usually, the roots of the following Legendre polynomials are $x_1, \dots, x_n$:

a. $p_2(x) = x^2 - \frac{1}{3}$, b. $p_3(x) = x^3 - \frac{3}{5}x$, c. $p_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$, etc.

Then use $x_1, \dots, x_n$ into Lagrange's polynomial to find $c_1 \sim c_n$:

$$f(x) = \sum_{i=1}^n f(x_i) L_{n,i}(x)$$

$$\int_{-1}^1 f(x) dx = \sum_{i=1}^n f(x_i) \int_{-1}^1 L_{n,i}(x) dx = \sum_{i=1}^n c_i f(x_i)$$

$$c_i = \int_{-1}^1 L_{n,i}(x) dx.$$

Remarks:

1. $n=2$, $x_1 = -x_2 = -0.577$, $c_1 = c_2 = 1$.
2. $n=3$, $x_1 = -x_3 = -0.775$, $x_2 = 0$, $c_1 = c_3 = 0.556$, $c_2 = 0.889$
3. $n=4$, $x_1 = -x_4 = -0.861$, $x_2 = -x_3 = 0.340$, $c_1 = c_4 = 0.348$, $c_2 = c_3 = 0.652$

Consider $\int_a^b f(x) dx$

Let $\frac{x-a}{b-a} = \frac{\eta - (-1)}{1 - (-1)}$ or $x = \eta \frac{b-a}{2} + \frac{1}{2}(b+a)$

Then $\int_a^b f(x) dx = \frac{b-a}{2} \int_{-1}^1 f\left[\frac{a+b}{2} + \frac{b-a}{2}\eta\right] d\eta$

$$\int_a^b f(x) dx = \frac{b-a}{2} \sum_{i=1}^n c_i f\left[\frac{a+b}{2} + \frac{b-a}{2}\eta_i\right]$$

§ Multiple integral

A. $\int_c^d \int_a^b f(x, y) dx dy$

a. Simpson's rule

$$x_0 = a, \quad x_{2n} = b, \quad y_0 = c, \quad y_{2m} = d, \quad h = (b-a)/2n, \quad k = (d-c)/2m,$$

$$x_i = a + ih, \quad y_j = c + jk$$

$$\begin{aligned} \int_c^d \int_a^b f(x, y) dx dy &= \frac{h}{3} \int_c^d [f(x_0, y) + 2 \sum_{i=2}^n f(x_{2i-2}, y) + 4 \sum_{i=1}^n f(x_{2i-1}, y) + f(x_{2n}, y)] dy \\ &= \frac{hk}{9} \{ [f(x_0, y_0) + 2 \sum_{j=2}^m f(x_0, y_{2j-2}) + 4 \sum_{j=1}^m f(x_0, y_{2j-1}) + f(x_0, y_{2m})] \\ &\quad + 2 \sum_{i=2}^n [f(x_{2i-2}, y_0) + 2 \sum_{j=2}^m f(x_{2i-2}, y_{2j-2}) + 4 \sum_{j=1}^m f(x_{2i-2}, y_{2j-1}) + f(x_{2i-2}, y_{2m})] \} \end{aligned}$$

$$\begin{aligned}
& +4\sum_{i=1}^n [f(x_{2i-1}, y_0) + 2\sum_{j=2}^m f(x_{2i-1}, y_{2j-2}) + 4\sum_{j=1}^m f(x_{2i-1}, y_{2j-1}) + f(x_{2i-1}, y_{2m})] \\
& +[f(x_{2n}, y_0) + 2\sum_{j=2}^m f(x_{2n}, y_{2j-2}) + 4\sum_{j=1}^m f(x_{2n}, y_{2j-1}) + f(x_{2n}, y_{2m})] \}
\end{aligned}$$

b. Gaussian Quadrature

$$\frac{x-a}{b-a} = \frac{\xi - (-1)}{1 - (-1)}, \quad \frac{y-c}{d-c} = \frac{\eta - (-1)}{1 - (-1)} \quad \text{or}$$

$$x = \frac{b-a}{2}\xi + \frac{b+a}{2}, \quad y = \frac{d-c}{2}\eta + \frac{d+c}{2}$$

$$\begin{aligned}
\int_c^d \int_a^b f(x, y) dx dy &= \frac{(b-a)(d-c)}{4} \int_{-1}^1 \int_{-1}^1 f\left[\frac{a+b}{2} + \frac{b-a}{2}\xi, \frac{c+d}{2} + \frac{d-c}{2}\eta\right] d\xi d\eta \\
&= \frac{(b-a)(d-c)}{4} \sum_{i=1}^n \sum_{j=1}^m c_i c_j f\left[\frac{a+b}{2} + \frac{b-a}{2}\xi_i, \frac{c+d}{2} + \frac{d-c}{2}\eta_j\right]
\end{aligned}$$

$$B. \int_a^b \int_{y=g_1(x)}^{y=g_2(x)} f(x, y) dy dx$$

a. Simpson's rule

$$y_0 = g_1(x), \quad y_{2m} = g_2(x), \quad k(x) = (g_2(x) - g_1(x))/2m, \quad y_j = g_1(x) + jk(x).$$

$$\begin{aligned}
\int_a^b \int_{y=g_1(x)}^{y=g_2(x)} f(x, y) dy dx &= \int_a^b \frac{h}{3} [f(x, y) + 2\sum_{i=2}^n f(x_{2i-2}, y) + 4\sum_{i=1}^n f(x_{2i-1}, y) + f(x_{2n}, y)] dx \\
&= \int_a^b \frac{k(x)}{3} [f(x, y_0) + 2\sum_{j=2}^m f(x, y_{2j-2}) + 4\sum_{j=1}^m f(x, y_{2j-1}) + f(x, y_{2m})] dx
\end{aligned}$$

$$x_0 = a, \quad x_{2n} = b, \quad h = (b-a)/2n, \quad x_i = x_0 + ih, \quad k_i = k(x_i)$$

$$\begin{aligned}
& \int_a^b \int_{y=g_1(x)}^{y=g_2(x)} f(x, y) dy dx \\
&= \frac{h}{9} [k_0 f(x_0, y_0) + 2\sum_{i=2}^n k_{2i-2} f(x_{2i-2}, y_0) + 4\sum_{i=1}^n k_{2i-1} f(x_{2i-1}, y_0) + k_{2n} f(x_{2n}, y_0)] \\
&+ \frac{2h}{9} [k_0 \sum_{j=2}^m f(x_0, y_{2j-2}) + 2\sum_{i=2}^n k_{2i-2} \left(\sum_{j=2}^m f(x_{2i-2}, y_{2j-2}) \right) + 4\sum_{i=1}^n k_{2i-1} \left(\sum_{j=2}^m f(x_{2i-1}, y_{2j-2}) \right) \\
&+ k_{2n} \left(\sum_{j=2}^m f(x_{2n}, y_{2j-2}) \right)] \\
&+ \frac{4h}{9} [k_0 \sum_{j=1}^m f(x_0, y_{2j-1}) + 2\sum_{i=2}^n k_{2i-2} \left(\sum_{j=1}^m f(x_{2i-2}, y_{2j-1}) \right) + 4\sum_{i=1}^n k_{2i-1} \left(\sum_{j=1}^m f(x_{2i-1}, y_{2j-1}) \right) \\
&+ k_{2n} \left(\sum_{j=1}^m f(x_{2n}, y_{2j-1}) \right)]
\end{aligned}$$

$$+k_{2n}\left(\sum_{j=1}^m f(x_{2n}, y_{2j-1})\right)]$$

$$+\frac{h}{9}[k_0 f(x_0, y_{2m})+2\sum_{i=2}^n k_{2i-2} f(x_{2i-2}, y_{2m})+4\sum_{i=1}^n k_{2i-1} f(x_{2i-1}, y_{2m})+k_{2n} f(x_{2n}, y_{2m})]$$

b. Gaussian quadrature

$$\int_{y=g_1(x)}^{y=g_2(x)} f(x, y) dy =$$

$$\frac{y-g_1(x)}{g_2(x)-g_1(x)} = \frac{\eta+1}{2} \quad \text{or} \quad y = \frac{g_2(x)-g_1(x)}{2}\eta + \frac{g_2(x)+g_1(x)}{2}$$

$$\int_{y=g_1(x)}^{y=g_2(x)} f(x, y) dy = \frac{g_2(x)-g_1(x)}{2} \int_{-1}^1 f\left[x, \frac{g_2(x)-g_1(x)}{2}\eta + \frac{g_2(x)+g_1(x)}{2}\right] d\eta$$

$$= \sum_{j=1}^m c_j \left\{ \frac{g_2(x)-g_1(x)}{2} f\left[x, \frac{g_2(x)-g_1(x)}{2}\eta_j + \frac{g_2(x)+g_1(x)}{2}\right] \right\}$$

$$\int_a^b \int_{y=g_1(x)}^{y=g_2(x)} f(x, y) dy dx =$$

$$\sum_{j=1}^m c_j \int_a^b \left\{ \frac{g_2(x)-g_1(x)}{2} \right\} f\left[x, \frac{g_2(x)-g_1(x)}{2}\eta_j + \frac{g_2(x)+g_1(x)}{2}\right] dx$$

$$x = \frac{b-a}{2}\xi + \frac{a+b}{2}$$

$$G_1(\xi) = \frac{1}{2} \left\{ g_2\left(\frac{a+b}{2} + \frac{b-a}{2}\xi\right) - g_1\left(\frac{a+b}{2} + \frac{b-a}{2}\xi\right) \right\}$$

$$G_2(\xi) = \frac{1}{2} \left\{ g_2\left(\frac{a+b}{2} + \frac{b-a}{2}\xi\right) + g_1\left(\frac{a+b}{2} + \frac{b-a}{2}\xi\right) \right\}$$

$$\sum_{j=1}^m c_j \int_a^b \left\{ \frac{g_2(x)-g_1(x)}{2} \right\} f\left[x, \frac{g_2(x)-g_1(x)}{2}\eta_j + \frac{g_2(x)+g_1(x)}{2}\right] dx$$

$$= \frac{b-a}{2} \sum_{j=1}^m c_j \int_{-1}^1 G_1(\xi) f\left[\frac{b-a}{2}\xi + \frac{b+a}{2}, G_1(\xi)\eta_j + G_2(\xi)\right] d\xi$$

$$= \frac{b-a}{2} \sum_{i=1}^n c_i \sum_{j=1}^m c_j G_1(\xi_i) f\left[\frac{b-a}{2}\xi_i + \frac{b+a}{2}, G_1(\xi_i)\eta_j + G_2(\xi_i)\right]$$

§ Improper integral

Let $f(x) = \frac{g(x)}{(x-a)^p}$, $0 < p < 1$, $g(x)$ is continuous, then $\int_a^b f(x) dx$ exists

To compute $\int_a^b \frac{g(x)}{(x-a)^p} dx$,

We use

$$p_4(x) = g(a) + g'(a)(x-a) + \frac{1}{2}g''(a)(x-a)^2 + \frac{1}{3!}g'''(a)(x-a)^3 + \frac{1}{4!}g^{(iv)}(a)(x-a)^4$$

$$g(x) = g(x) - p_4(x) + p_4(x)$$

$$\int_a^b \frac{g(x)}{(x-a)^p} dx = \int_a^b \frac{g(x) - p_4(x)}{(x-a)^p} dx + \int_a^b \frac{p_4(x)}{(x-a)^p} dx$$

where

$$\int_a^b \frac{p_4(x)}{(x-a)^p} dx = \sum_{k=0}^4 \int_a^b \frac{g^{(k)}(a)}{k!} (x-a)^{k-p} dx = \sum_{k=0}^4 \frac{g^{(k)}(a)}{k!(k+1-p)!} (b-a)^{k+1-p},$$

And

$$\int_a^b \frac{g(x) - p_4(x)}{(x-a)^p} dx = \int_a^b G(x) dx$$

$$\text{where } G(x) = \begin{cases} \frac{g(x) - p_4(x)}{(x-a)^p}, & a < x \leq b \\ 0, & x = a \end{cases}$$

HW4

1. Determine the values $\int_1^2 e^x \sin(4x) dx$ with $h = 0.1$ by
 - a. Use the composite trapezoidal rule
 - b. Use the composite Simpsons' method
 - c. Use the composite midpoint rule
2. Approximate $\int_1^{1.5} x^2 \ln x dx$ using Gaussian Quadrature with $n = 3$ and $n = 4$. Then compare the result to the exact value of the integral.
3. Approximate $\int_0^{\pi/4} \int_{\sin x}^{\cos x} (2y \sin x + \cos^2 x) dy dx$ using
 - a. Simpson's rule for $n = 4$ and $m = 4$
 - b. Gaussian Quadrature, $n = 3$ and $m = 3$
 - c. Compare these results with the exact value.
4. Use the composite Simpson's rule and $n = 4$ to approximate the improper integral a) $\int_0^1 x^{-1/4} \sin x dx$, b) $\int_1^\infty x^{-4} \sin x dx$ by use the transform $t = x^{-1}$